Available online at www.sciencedirect.com

SCIENCE DIRECT®

Theoretical Computer Science 349 (2005) 443–451

Theoretical Computer Science

www.elsevier.com/locate/tcs

Conway's Angel in three dimensions ☆

Martin Kutz*, [1]

Max-Planck Institut für Informatik, Im Stadtwald, 66123 Saarbrücken, Germany

Received 22 December 2004; received in revised form 20 July 2005; accepted 18 August 2005

Communicated by A. Fraenkel

Abstract

The Angel–Devil game is an infinite game played on an infinite chess board: In each move the Angel, a generalized chess king, jumps from his current square to some location at distance at most k, while his opponent, the Devil, blocks squares trying to strand the Angel. The Angel wins if he manages to fly on forever. It is a long-standing open question whether some Angel of sufficiently large power k can escape.

We show that in the three-dimensional analog of the game the 13-Angel can win. Our proof is constructive and provides an explicit infinite escape strategy.

Keywords: Angel and Devil; Combinatorial games; Infinite games

1. Introduction

Two players, *the Angel* and *the Devil*, play a game on an infinite chess board. The Angel is an actual figure moving across the board like some chess piece, while his opponent does not live on the board but only manipulates it. In each move, the Devil blocks an arbitrary square of the board such that this location may no longer be stepped upon by the Angel. The Angel in turn flies in each move from his current position, indexed by $(x, y) \in \mathbb{Z}^2$, say, to some unblocked square at distance at most k, for some fixed integer k, i.e., to some position $(x', y') \neq (x, y)$ with $|x' - x|, |y' - y| \leqslant k$. Note that Devil moves are not restricted to the Angel's proximity or limited by any other distance bounds; he can pick squares at completely arbitrary locations.

The Devil wins if he can stop the Angel, that is, if he manages to get him in a position with all squares in the $(2k+1) \times (2k+1)$ area around him blocked. The Angel wins simply if he succeeds to fly on forever. The open question is, whether for some sufficiently large integer k the Angel with distance bound k, called the *k-Angel*, can win this game.

First variants of this game were discussed by Gardner [4], who names Silverman and Epstein as original inventors. In it's present form the Angel game first appeared in Berlekamp et al. classic [1, Chapter 19]. Amongst detailed analyses of games with kings and other chess pieces on finite boards against Devils with certain additional restrictions,

☆ This work was previously published as part of the author's Ph.D. Thesis [5].

* Tel.: +49 681 9325 119; fax: +49 681 9325 199.

E-mail address: mkutz@mpi-inf.mpg.de.

[1] The research was done while the author was member of the European graduate program "Combinatorics, Geometry, and Computation" supported by the Deutsche Forschungsgemeinschaft (GRK 588/2).

0304-3975/$ - see front matter © 2005 Published by Elsevier B.V.
doi:10.1016/j.tcs.2005.08.034

the authors coin the names "Angel" and "Devil" for the two competitors and give a thorough proof that the chess king can be caught on a 33-by-33 board. Then Conway [3] focused entirely on the infinite Angel game, trying to explain possible pitfalls with certain natural escape attempts and pointing out the hardness of the problem. The role he played in popularizing the game eventually led to the naming "Conway's Angel" and we like to stick with this tradition.

Besides all variants, the central open question remains whether some Angel of sufficient power can escape forever.

The three-dimensional board. In this paper we consider the three-dimensional analog of the original problem and show that the extra degree of freedom allows already an Angel of moderate power to escape forever.

Formally, a 3D-Angel lives in a three-dimensional world of cubes, indexed by coordinates in $\mathbb{Z}^3$. Like in the plane, in each move the k-Angel jumps from his current position (x, y, z) to some other cube (x', y', z') with $|x'-x|$, $|y'-y|$, $|z'-z| \leqslant k$ and in turn, the Devil blocks some cube of his choice. We prove the following.

Theorem 1. *On the three-dimensional board the* 13*-Angel can escape forever*.

The three-dimensional problem has only been mentioned once in the literature, namely in [1], where the authors actually report that escape strategies for "many-dimensional Angels" are known. However, the respective proof, attributed to Körner, has never been published.

Independently from this work, Béla Bollobás and Imre Leader [2] have also found a proof that in three dimensions the Angel can escape.

2. A hierarchy of boxes

Our escape strategy divides the world into an infinite hierarchy of larger and larger boxes. The Angel will have to make sure that on each level, his current box contains not too many Devil blocks. This shall then guarantee his free travel.

A remark on terminology. Our usage of the word "cube" might get a little confusing when we speak about our hierarchy, since higher-level boxes will themselves be cubes—of cubes of cubes of cubes, etc. We shall use the expression *elementary cube* to emphasize that we mean the basic locations of the board, while the term *box* be reserved for collections of such objects. With other expressions the intended meaning should in general be clear from the context.

On the first level, the world is regularly partitioned into boxes of sidelength 13, such that the origin $0 \in \mathbb{Z}^3$, where the Angel starts, lies at the very center of one of these boxes. Formally, the first level H_1 is the collection of all boxes

$$\begin{aligned} H_1^{(u,v,w)} := \{(x, y, z) \in \mathbb{Z}^3 \mid\ & 13u - 6 \leqslant x \leqslant 13u + 6, \\ & 13v - 6 \leqslant y \leqslant 13v + 6, \\ & 13w - 6 \leqslant z \leqslant 13w + 6\}, \end{aligned}$$

with $u, v, w \in \mathbb{Z}$, where we reference elementary cubes of the world via their coordinates $(x, y, z) \in \mathbb{Z}^3$.

The sidelength 13 corresponds to the power of the 13-Angel. From level 2 on, sidelengths grow by a factor of 29 per step, where there is no deeper reason for the choice of this particular value except that it makes the forthcoming computations work. On each level we again demand that the origin lie at the very center of the one box that contains it. Technically, for $j \geqslant 2$ the jth level H_j of our hierarchy is the collection of all boxes

$$\begin{aligned} H_j^{(u,v,w)} := \{H_{j-1}^{(a,b,c)} \mid\ & 29u - 14 \leqslant a \leqslant 29u + 14, \\ & 29v - 14 \leqslant b \leqslant 29v + 14, \\ & 29w - 14 \leqslant c \leqslant 29w + 14\}, \end{aligned}$$

with $u, v, w \in \mathbb{Z}$.

So any box on level $j \geqslant 2$ contains 29^3 boxes on level $j-1$ and the whole hierarchy is symmetric to the origin. Note that formally the elements of a higher-level box are again boxes, which is what we want. But with a certain laxness we shall also consider a level-j box simply as the set of the $(13 \cdot 29^{j-1})^3$ elementary cubes that lie inside it. In this vein we define the *level-j box of a cube $a \in \mathbb{Z}^3$* to be the unique box in H_j that "contains" the elementary cube a and

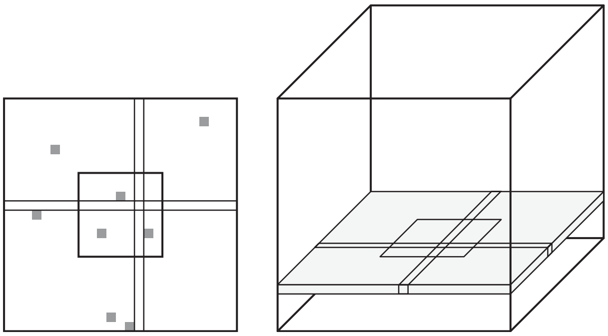

Fig. 1. Clear positions.

denote it by

$$Q_j(a).$$

Further we define a mass function μ for all boxes A on all levels of our hierarchy, letting

$$\mu(A)$$

count the number of elementary cubes inside A that have already been blocked.

2.1. Clear roads ahead

Globally, the Angel's route through our hierarchy of boxes will be guided by simple mass constraints, in a quite simple way. The basic step, the transition between two adjacent boxes, however, requires some dirty work. We need to introduce a few technical notions to ensure that locally the Angel does not get stuck in unfortunate arrangements of blocks.

Definition 1. Let E be a quadratic grid of 29×29 cubes with some cubes marked *forbidden*. We say that a cube q of E *lies clear in* E if

- no more than 12 of the $29^2 = 841$ cubes in E are forbidden,
- q lies in the central 13-by-13 square of E,[2] and
- the two axis-parallel lines through q in E contain no forbidden points.

The left-hand side of Fig. 1 displays such a pair of orthogonal lines that meet in the central 13-by-13 region and are free of forbidden cubes.

Let C be a cubic grid of $29 \times 29 \times 29$ cubes with some cubes marked *forbidden*. We say that a cube q of C *lies clear in* C if

- no more than 333 of the $29^3 = 24,389$ cubes in C are forbidden and
- q lies clear in one of the three axis-parallel 29×29 planes through q in C.

See the cube in Fig. 1.

The idea behind the above definitions is, as we said before, to guarantee free navigation from a clear cube within a sidelength-29 box to somewhere outside this box. A cube that lies clear will have enough free space around it to guarantee an easy route out. The forbidden cubes may, of course, not be used for travel. We do not speak of blocked cubes in Definition 1 because the little cubes will usually themselves be boxes of smaller cubes. Yet, forbidden cubes will be *almost* blocked, meaning that their mass exceeds a certain threshold.

---

[2] The occurrence of the number 13 here is coincidental. This is a "different" 13 than the one from Theorem 1.

For *paths* through such boxes we allow axis-parallel steps of unit distance only. That is, a single step of a path is a change of ± 1 in just one coordinate. So box transitions are measured in the 1-norm although basic Angel moves are restricted by the infinity norm. This restriction is due to the hierarchical structure of our argument. The Angel will be able to travel between two little cubes inside the big cube in Definition 1 only if these cubes share a face which may be used for a transition on the next lower level.

From a purist's point of view, the grids E and C of Definition 1 could, of course, just be called grid graphs, with "cubes" replaced by "vertices." Then a path would just be a path in the graph-theoretic sense and the following lemmas are in fact just statements about such grid graphs. However, we like to keep with our view of cubes and boxes in order to emphasize the actual purpose of the above definitions.

The following lemma about planes only serves as a tool for the three-dimensional case. Our actual interest will be in paths through boxes.

Lemma 2. *Let q be a cube lying clear in a 29×29 grid E. Then at least $763 = 29^2 - 78$ cubes of E are reachable from q in at most 40 steps each.*

Proof. Any cube on the two lines through q is by assumption reachable directly through that respective line. For every other point $p \in E$ we consider the two potential paths that run parallel to the axes with exactly one turn. A cube p may not be reachable on either of these two paths for two reasons: both paths are blocked or p is a forbidden cube itself. Since by the special choice of our paths, a single pair of forbidden cubes covers at most one cube of E, the first situation can happen for at most $\binom{12}{2} = 66$ cubes, the second, by definition, for at most 12; which makes 78 inaccessible places altogether. One easily computes that any of the remaining $29^2 - 78 = 763$ cubes is reachable in at most 40 steps since the distance from any location in the central region to any side of E is at most 20. $\square$

Lemma 3. *Let q be a cube lying clear in a $29 \times 29 \times 29$ box grid C and let D be another $29 \times 29 \times 29$ box aligned with C along one of its faces, also with no more than 333 points marked forbidden. Then there exists a cube r lying clear in D such that there is a path of length at most 165 from q to r, which after the first 96 steps uses no more cubes in C.*

Proof. Let E denote a plane within C in which q lies clear as required by Definition 1. The basic idea for the path construction is to pick a suitable plane F in D, which will contain the target point r, and then to find many disjoint paths from E to F not all of which can be blocked by forbidden cubes.

Observe that by the pigeon-hole principle, among the 29 axis-parallel planes in D that lie parallel to that face of D which borders on C, at least one contains no more than 12 forbidden cubes ($29 \cdot 13 = 377 > 333$). Choose F to be such a plane. For both dimensions of F, at most 12 of the 13 axis-parallel lines passing through the central 13×13 region of F are blocked by forbidden cubes, which leaves at least one clear line in each direction. We choose r as the intersection of two such lines, which makes it lie clear in D. We now distinguish two different cases: when the planes E and F are parallel and when they are not.

Case 1: E parallel to F. Partition the union of C and D into the $29^2 = 841$ disjoint lines that intersect E and F orthogonally. By Lemma 2, all but 78 of these lines intersect E in cubes that are reachable from q in at most 40 steps and likewise, all but 78 lines intersect F in cubes that are reachable in 40 steps from r. This leaves $841 - 2 \cdot 78 = 685$ lines whose intersections with E and F are reachable in 40 steps from q, respectively, r. By assumption, there are no more than 666 forbidden cubes in C and D altogether, so several of those lines are completely free. Since the distance between the planes E and F is bounded by twice the sidelength of the boxes C and D, we get a path from q to r of no more than $2 \cdot (40 + 29) - 1 = 137$ steps.

Case 2: E and F are not parallel. It can be treated similarly. Only the connecting lines must be chosen in a more complicated way. Partition the union of C and D into 29 parallel planes of size 29×58 such that each plane intersects E and F in exactly one line. Within each of these planes we match the 29 cubes of C with the 29 cubes of D by 29 disjoint paths as displayed in Fig. 2. As in the first case, we thus get a positive amount of paths connecting locations in E reachable from q to locations in F reachable from r, that are all free of forbidden cubes. The length bound is a little worse, however. Paths in Fig. 2 can require up to $28 + 29 + 28 = 85$ steps, which together with the paths within the planes E and F yields an upper bound of 165 steps from q to r. It is easily checked that in either configuration we spend no more than 96 steps inside C. $\square$

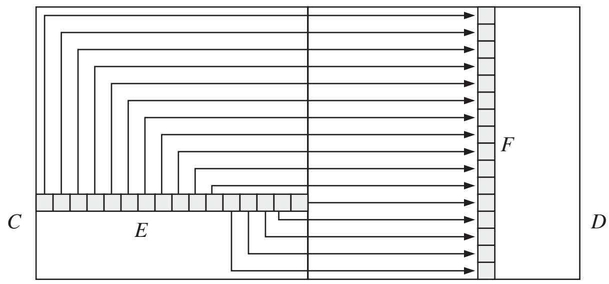

Fig. 2. Traveling between non-parallel planes E and F.

2.2. Clear boxes

We want to apply the box-travel lemma to boxes of our hierarchy (H_j). Therefore we have to define which level-$(j-1)$ subboxes inside a level-j box should be considered forbidden. This shall, for now, depend on a simple mass constraint. (Later we will also need a slightly modified definition.)

Definition 2. Call a box $A' \in H_{j-1}$, $j \geqslant 2$, *light* if

$$\mu(A') \leqslant \tfrac{17}{3} \cdot 165^{j-1} \tag{1}$$

and *heavy* otherwise.[3] We then say that the Angel's position a is *nice on level j* if the subbox $Q_{j-1}(a)$ lies clear in $Q_j(a)$, with exactly the heavy level-$(j-1)$ boxes forbidden. The position is *nice on level* 1 simply if

$$\mu(Q_1(a)) \leqslant 1157. \tag{2}$$

We say that a position is *nice up to level j* if it is nice on all levels from 1 through j.

The notion of niceness will be suitable to guarantee an escape route out of the current level-j box $Q_j(a)$. Recall that the constant 165 is exactly the step bound provided by Lemma 3. Level 1 receives a special treatment because it forms the induction basis, founding our hierarchy argument on actual Angel moves.

3. The main induction—escaping from larger and larger boxes

With the notion of niceness at hand, it is actually rather straightforward to formulate an appropriate induction hypothesis for Angel strategies that allow to travel between arbitrarily large boxes. Only a few constants remain to be chosen thoroughly. And of course, we have to make some assumption on the target box we want to run into. Actually, a simple mass constraint will do.

Proposition 1. *Let B be one of the six level-j boxes neighboring the Angel's current box $A \in H_j$, $j \geqslant 1$. If his current position is nice up to level j and the mass of B is bounded by*

$$\mu(B) \leqslant 7 \cdot 165^{j}, \tag{3}$$

then the 13*-Angel can get in no more than*

$$2 \cdot 165^{j-1} \tag{4}$$

elementary moves from his actual position in A to some location in B such that after he has arrived there, his position will be nice up to level j again.

[3] We prefer to write $j-1$ instead of simply j to emphasize that although lightness is a property of a single box, it shall always be used in reference to the containing box on level j.

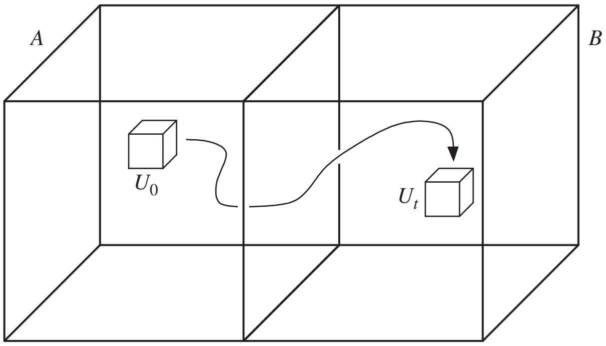

Fig. 3. A single induction step.

Note that the coefficient 7 in (3) is slightly larger than the 17/3 in (1). So for the box B in Proposition 1, we impose a weaker mass constraint than would be required for being considered light as a subbox of the containing box on level $j+1$. We also remark that 165^j lies somewhere in between the sidelength of a level-j box and the number of points in a face of such a box. One could say that with increasing level, the mass bound (3) grows strictly faster than one-dimensional objects but strictly slower than two-dimensional objects. Likewise the path length (4); compared to the diameter of a level-j box, it gets arbitrarily large, hence, seen from a far distance, the Angel slows down to almost zero speed. Compared to surface growth, however, and this is the crucial measure because potential Devil obstacles must be two-dimensional, the speed can actually be seen to *increase* by $29^2/165 > 5$ per level.

Proof of Proposition 1. By induction on j. The induction basis is $j = 1$. We have exactly 2 moves to get from the current sidelength-13 box A to an arbitrary elementary cube in B. By niceness, A contains at most 1157 Devil blocks and by (3), B contains no more than $7 \cdot 165 = 1155$ blocks. Thus by the pigeon-hole principle, any 7 planes within the current box A or the target box B contain at least $7 \cdot 13^2 - 1157 = 26$ free locations. Hence, the 13-Angel may jump from its current position a to some other elementary cube in A at most 7 units away from B. From there he can reach in just one further jump any point within the first 7 layers of B, which still contain some unblocked cubes. He jumps to one of them with his second move. The two Devil answers cannot raise the mass of B above 1157, so afterwards the position will be nice on level-1 again, as required.

Induction step from $j-1$ to $j \geqslant 2$. Niceness of the current position a guarantees that there are at most 333 heavy subboxes A' in A, all the other boxes satisfying the lightness condition (1). In our target box B we also mark forbidden subboxes, based however, on a slightly stronger mass constraint. Mark a level-$(j-1)$ subbox B' in B forbidden if it does *not* satisfy

$$\mu(B') \leqslant \tfrac{11}{3} \cdot 165^{j-1}. \tag{5}$$

So in B, non-forbidden subboxes are "ultra light" (compare Definition 2). Since 334 such forbidden boxes in B would yield a total mass of

$$334 \cdot \tfrac{11}{3} \cdot 165^{j-1} > 7 \cdot 165^j,$$

our assumption (3) implies that B contains no more than 333 forbidden boxes, either.

Now there are two adjacent level-j boxes A and B with at most 666 level-$(j-1)$ subboxes forbidden altogether, based on two slightly different criteria. By niceness on level j of the current elementary cube a, the box $Q_{j-1}(a)$ lies clear within the box $A = Q_j(a)$. Further, the neighboring level-j box B contains fewer than 333 forbidden boxes. Hence, Lemma 3 applies to A and B, giving a path $(U_0, U_1, \ldots, U_t)$ of level-$(j-1)$ boxes with $t \leqslant 165$, from the current box $Q_{j-1}(a) = U_0$ to some U_t that lies clear in B with respect to the ultra-light boxes there. Moreover, the lemma guarantees that from U_{97} on all boxes lie in B (see Fig. 3).

We use this path of boxes to obtain an actual strategy that gets the Angel from a to some point in U_t. Niceness up to level j at his starting position a implies niceness up to level $j-1$, so we apply our induction hypothesis on level $(j-1)$ to the pair U_0, U_1, getting the Angel to a position within U_1 that is also nice up to level $j-1$ and from there to a nice position inside U_2—and so on, all the way to some elementary cube b that is nice up to level j in U_t. However, this will only work if the mass constraint (3) is satisfied for the target box U_τ in each single transition between two adjacent boxes $U_{\tau-1}$ and U_τ.

This is easily checked. The whole journey from a to b would grant the Devil at most

$$165 \cdot 2 \cdot 165^{j-2} = 2 \cdot 165^{j-1} \tag{6}$$

moves. Even if he spends all of them on a single box U_τ in B, the mass of this box will remain bounded by

$$\mu(U_\tau) \leqslant \tfrac{11}{3} \cdot 165^{j-1} + 2 \cdot 165^{j-1} = \tfrac{17}{3} \cdot 165^{j-1}. \tag{7}$$

For a box U_τ that lies in A, we even know that it cannot receive more than $95 \cdot 2 \cdot 165^{j-2}$ Devil moves before we want to enter it, so that by the time we invoke Lemma 3 the following mass bound will hold:

$$\mu(U_\tau) \leqslant \tfrac{17}{3} \cdot 165^{j-1} + 95 \cdot 2 \cdot 165^{j-2} < 7 \cdot 165^{j-1}. \tag{8}$$

Both bounds, (7) and (8), satisfy requirement (3) of Proposition 1 with j replaced by the appropriate level $j-1$ there. Hence, all those transitions between the U_τ will be possible. Also note that the number of moves counted in (6) is exactly what we had to show for (4).

Eventually, the Angel reaches an elementary cube b in U_t in the required number of elementary moves such that by that time the resulting position is nice up to level $j-1$. It remains to show niceness on level j. To see this, recall that the relaxed mass bound for the originally ultra-light subboxes in B, which we computed in (7), matches exactly our definition (1) of light boxes. Hence, all subboxes B' of B that are heavy after the Angel's trip from a to b, had already been forbidden in the beginning when the box-travel lemma was invoked, and thus the terminal box U_t lies clear in B with respect to those boxes. In other words, b is nice on level j, too. □

4. From finite to infinite games

Proposition 1 almost immediately implies Theorem 1, by a standard compactness argument. For the formal proof, we first show that successful Devil strategies are always finite.

Lemma 4. *Every winning strategy for the Devil wins after a bounded number of moves. That is, it cannot be that the Angel is bound to lose but is able to delay his defeat for an arbitrarily long time.*

Proof. Consider the game tree of all possible plays under an assumed winning Devil strategy σ. Its leaves are exactly those positions in which the Angel cannot move anymore and thus has lost. This tree has a bounded number of options at each Angel node (no more than $(2k+1)^3$) and just one option at each Devil node, namely the one prescribed by σ; and because σ is a winning strategy, it contains no infinite paths. Therefore, König's lemma implies that this tree has finite depth, d, say. This means that the strategy σ allows no more than d moves before the Angel is stuck, independent of how the Angel plays. □

Proof of Theorem 1 (*non-constructive version*). At the very beginning of the game, all boxes on all levels of our hierarchy are empty and thus light within their respective containing boxes. By the symmetry of the hierarchy, the Angel starts at the very center of the box $Q_j(0)$ on every level $j \geqslant 1$. Therefore the starting position is nice on every level $j \geqslant 1$.

By Proposition 1, the Angel can thus travel to some adjacent box on any previously given level of the hierarchy, which allows him to escape the Devil for any previously chosen amount of time. So the Devil cannot have a strategy that always catches the Angel after a fixed number of moves and thus, by Lemma 4, he does not have a winning strategy at all.

This guarantees a winning strategy for the Angel: at each turn he can move such that the Devil does not get a winning strategy for the resulting position. So by induction, the Angel can run on forever. □

4.1. An explicit infinite strategy

The preceding argument contained purely existential steps so that the proof does not tell us how the Angel should actually play to escape forever. We now provide a second, constructive proof for Theorem 1. It is in some sense simpler than the first one because it avoids the issue of Lemma 4 and the consideration of infinite paths, but it has the drawback that we cannot use Proposition 1 as a black box anymore but have to revisit some details from its proof. The subsequent argumentation relies on that particular Angel strategy and might not work for possible variants or improvements of Proposition 1.

Proof of Theorem 1 (*constructive version*). We start escape strategies on *all* levels of the hierarchy simultaneously; in such a way that on initial segments those strategies are compatible. Therefore we introduce a small technical convention about the paths provided by Lemma 3.

Unrolling the induction in the proof of Proposition 1, we can interpret that result as a concrete strategy for journeys between adjacent boxes of our hierarchy, which on each level invokes Lemma 3 as an algorithm (implicitly given by its proof) for path finding in grid graphs. In this algorithmic view, let us agree that whenever Lemma 3 is used to find a path between two boxes that contain no forbidden cubes at all, it returns a path that starts with a step in the direction of the target box.

The Angel begins by traveling from the origin 0 to a nice position a_1 in the level-1 box B_1 that lies directly behind (in positive z-direction, say) the initial box $Q_1(0)$. Having arrived at position a_1, we can interpret these first steps as the initial sequence of a travel from the box $Q_2(0)$ to a nice position a_2 in the level-1 box B_2 just behind the initial level-2 box $Q_2(0)$. As we already observed in the non-constructive proof above, such a strategy exists by Proposition 1 and by our convention it would have started with a travel to a position in B_1, just as we did. We now follow the new level-2 strategy until we reach the position a_2. At that point, we again interpret this journey as the initial sequence of a travel from the origin to a nice position a_3 in the level-3 box behind $Q_3(0)$. Iterating this argument indefinitely, we obtain an infinite escape strategy for the Angel. The crucial argument here is that what we have done up to some point, will always fit into strategies on higher levels that we have not considered yet. □

5. Why our hierarchy does not work in 2D

One might want to try to transform the hierarchy approach for the three-dimensional case into an escape strategy for the two-dimensional game. Such an attempt would face two major obstacles. First, as we already remarked after the statement of Proposition 1, the step bound (4) grows strictly faster than the sidelengths of the boxes. This effect is due to the detours that result from each application of Lemma 3. On higher and higher levels, the effective speed of the Angel thus gets arbitrarily slow. In the plane, this would allow the Devil to completely encircle the Angel on a sufficiently large scale since the boundary of a rectangle is proportional to the radius. Hence, we would need an improved path finder that might probably employ some means of charging Devil moves against Angel moves such that Devil plays that force the Angel to make detours cannot be counted for wall building far away.

But even if one should succeed in maintaining the "effective speed" of the Angel, there would remain a more fundamental problem about hierarchical strategies like the one we presented. While routing out of a level-j rectangle R (or whatever regular shape might be used) the Angel must at some point decide which of the subrectangles on level $j-1$ should be the last on the way out. Then he will have to pass through the outward side S' of this subrectangle R' at some time in the future. While the Angel approaches R', the Devil uses a certain number of his moves, proportional to the sidelength of R', to destroy points of S' at some density. After the Angel has entered R', he must then, as before, pick some subrectangle R'' of R' that should be the last before he leaves R' through S' and thereby confine himself to pass through its outward side $S'' \subset S'$, shown in Fig. 4. Again, the Devil uses a certain number of moves to increase the density on S'' *by the same amount* as on the previous level.

Repeated application of this scheme on sufficiently many levels eventually yields a completely blocked line through which the Angel would have to travel. In essence we just sketched a hierarchical version of Conway's Fool Theorem [3], which states that any Angel that tries to escape in one fixed direction can be caught. The implication for hierarchical approaches in the plane is clear: the different levels of an Angel's hierarchy will have to interact in a considerably more sophisticated way than is sufficient for an escape in space.

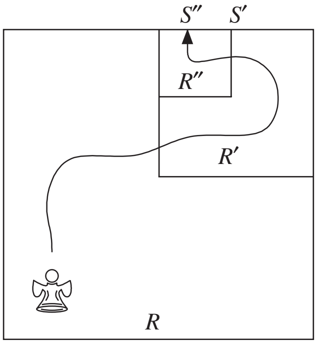

Fig. 4. A failing hierarchy-approach in 2D.

Acknowledgment

I thank Stefan Geschke for showing me the set-theoretic foundations of infinite games and for inspiring discussions about the Angel and the Devil.

References

[1] E.R. Berlekamp, J.H. Conway, R.K. Guy, Winning Ways for your Mathematical Plays, Vol. 2, Games in Particular, Academic Press, New York, 1982.
[2] B. Bollobás, I. Leader, The Devil and the Angel in three dimensions, J. Combin. Theory Ser. A, to appear.
[3] J.H. Conway. The Angel problem, in: R. Nowakowski (Ed.), Games of No Chance, Vol. 29, MSRI Publications, 1996, pp. 3–12.
[4] M. Gardner, Mathematical games, Sci. Amer. 230 (2) (1974) 106–108.
[5] M. Kutz, The Angel problem, positional games, and digraph roots, Ph.D. Thesis, Freie Universität Berlin, 2004. http://www.diss.fu-berlin.de/2004/250/indexe.html.